Random Matrix Theory: Lecture 1

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Wigner model

Let X_n random matrix, $X_n(i,j)$ i.i.d. $\sim N(0,1)$.

- Wigner (1955): $M_n = \frac{X_n + X_n^T}{2}$

Question: study

$$F_n(x) = \frac{\{\# \; i : \lambda_i^{(n)} \leq x\}}{n}$$

$$\nu_n(\bullet) = \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i}(\bullet),$$

where δ_x stands for the Dirac delta measure.

Recall $\lambda_1^k + \ldots + \lambda_n^k = trace(M_n^k)$.

By the law of large numbers

$$\begin{aligned}\int x\nu_n(dx) &= \frac{\lambda_1+\ldots+\lambda_n}{n} = \frac{1}{n}trace(M_n)\\ &= \frac{M_n(1,1)+\ldots+M_n(n,n)}{n} \to E(M_n(1,1)) = 0.\end{aligned}$$

and

$$\int x^2\nu_n(dx) = \frac{\lambda_1^2+\ldots+\lambda_n^2}{n} = \frac{1}{n}trace(M_n^2) = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} M_n(i,j)^2,$$

and for each i, a.s. $\frac{1}{n}\sum_{j=1}^{n} M_n(i,j)^2 \to 1,\ n\to\infty$,
thus $\int x^2\nu_n(dx) \to \infty$ as $n\to\infty$.

Expectation of the moments

We study $\frac{1}{\sqrt{n}}M_n$, then

$$\mu_n(\bullet) = \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i/\sqrt{n}}(\bullet),$$

Then $\int x^2 \mu_n(dx) \to 1$.
Define $m_k^{(n)} = \int x^k \mu_n(dx)$, and we calculate first $\lim_{n\to\infty} E(m_k^{(n)})$.
Using the trace identity

$$\begin{aligned} m_k^{(n)} &= \frac{1}{n}\left((\lambda_1/\sqrt{n})^k + \ldots + (\lambda_n/\sqrt{n})^k\right) = \frac{1}{n} trace\left((M_n/\sqrt{n})^k\right) \\ &= \frac{1}{n^{k/2+1}} \sum_{\bar{i}_k} P_n(\bar{i}_k), \end{aligned}$$

where, $\bar{i}_k = \{i_1, i_2, \ldots, i_k\}$, and

$$P_n(\bar{i}_k) = M_n(i_1, i_2) M_n(i_2, i_3) \ldots M_n(i_{k-1}, i_k) M_n(i_k, i_1).$$

If k odd, $E(P_n(\bar{i}_k)) = 0$. Then, study k even: $m_{2k}^{(n)}$

moments m_{2k}, digits $\leq k$

Two situations, when the set $\bar{i}_{2k}$ has exactly $k+1$ different digits, or when it has $\leq k$ digits: $m_{2k}^{(n)} = m_{2k}^{(n)}(I) + m_{2k}^{(n)}(II)$
In situation II there are at most n^k different ways to choose $\leq k$ digits, and with one selection of digits we can form at most $2k!$ arrangements for the pairs

$$(i_1, i_2)(i_2, i_3) \ldots (i_{2k-1}, i_k)(i_{2k}, i_1).$$

Moreover, recall that for h even, $E(M_n^h(i,j)) = (h-1)!!$, therefore there exist a constant α_k depending only on k such that

$$E(m_{2k}^{(n)}(II)) \leq \frac{1}{n^{k+1}} n^k \alpha_k,$$

which goes to 0 when $n \to \infty$ and k is fixed.

For situation I with $k+1$ digits in $\bar{i}_{2k}$, $E(P_n(\bar{i}_{2k}))$ is the product of k second moments $E(M_n^2(i,j)) = 1$.
Following the flow $i_1 \to i_2 \to \ldots i_{2k} \to i_1$ we notice that the set of pairs

$$(i_1, i_2)(i_2, i_3) \ldots (i_{2k-1}, i_k)(i_{2k}, i_1)$$

forms a *directed graph with no loops* made with $k+1$ *vertices* and $2k$ *edges*. By gluing the parallel edges it is formed a *tree* with no loops.
Having fixed one particular choice of these type of trees, there are $P(n, k+1)$ different permutations using $k+1$ digits from n numbers. Hence, we end up with

$$E(m_{2k}^{(n)}(I)) = \frac{1}{n^{k+1}} \frac{n!}{(n-(k+1))!} C_k,$$

C_k is the number of trees with no loops that one can form with $k+1$ vertices: $C_k = \frac{1}{k+1}\binom{2k}{k}$ the Catalan numbers.

Since $\frac{1}{n^{k+1}} \frac{n!}{(n-(k+1))!} = \frac{n}{n} \frac{n-1}{n} \dots \frac{n-k}{n}$ converges to 1 as $n \to \infty$,

$$\lim_{n\to\infty} E(m_k^{(n)}) = m_k = \begin{cases} 0 & k \text{ odd} \\ C_{k/2} & k \text{ even.} \end{cases}$$

Also

$$Var\left(m_k^{(n)}\right) \leq \frac{k}{n^{2+k}} n^k \alpha_k = \frac{c_k}{n^2} \to 0.$$

Then, by Borel-Cantelli lemma, almost surely $\lim_{n\to\infty} m_k^{(n)} = m_k$. The law for these moments is the semicircle law

$$\rho_{sc}(x) = \frac{1}{2\pi}\sqrt{4-x^2}, \; x \in [-2, 2].$$

Catalan numbers and Dyck paths

Given a tree of $k+1$ vertices and no loops, one can form a $2k-$steps trajectory p on the positive integers starting and finishing at 0 moving one unit at each step. : $p(i) \geq 0$ such that $p(0)=0, \; p(1)=1, \; p(2k)=0$ and $|p(i+1)-p(i)|=1$, these are called *Dyck* paths

The number of k-Dyck paths is

$$\binom{2k}{k} - \binom{2k}{k-1} = \frac{1}{k+1}\binom{2k}{k}.$$