Random Matrix Theory: Lecture 2

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Wishart ensemble

Wishart ensemble is $n \times n$ matrix given by $M_n = X_{m,n}^\top X_{m,n}$.
it also looks like

$$M_n v = \sum_{i=1}^{n} \langle q_i, v \rangle q_i.$$

$$q_i = \begin{pmatrix} X_n(i,1) \\ \vdots \\ X_n(i,n) \end{pmatrix}. \quad = i \text{ row (in colum form)}$$

Also $\gamma_n(i)$ is the i column of X_n.
Again, we are interested in the empirical spectral distribution ν_n.
Now the correct scaling is $\frac{1}{n} M_n$, i.e. $X_n(i,j) \sim N(0, n^{1/4})$.

Stieltjes transform

$$S_n(z) := \int_{-\infty}^{\infty} \frac{1}{x - z} \mu_n(dx), \;\; z \in \mathbb{C}.$$

for $z \notin \mathbb{R}$, the eigenvalues of $[M_n - zI]^{-1}$ are $\frac{1}{\lambda_1 - z}, \dots, \frac{1}{\lambda_n - z}$, then

$$S_n(z) = \frac{1}{n} trace \left([M_n - zI]^{-1} \right).$$

tricks

Suppose that A is a $n \times n$ symmetric matrix and B be its upper-left $(n-1) \times (n-1)$ minor submatrix

$$A = \begin{bmatrix} B & v \\ v^\top & a \end{bmatrix},$$

Then,

$$traceA^{-1} - traceB^{-1} = \frac{1 + v^\top B^{-2} v}{a - v^\top B^{-1} v}.$$

Now, suppose that C is the $n \times n$ matrix

$$C = \begin{bmatrix} B & 0 \\ 0 & z \end{bmatrix},$$

where $z \in \mathbb{C}$ and 0 represents a column or row with only zeros. Then,

$$traceC^{-1} = traceB^{-1} + \frac{1}{z}.$$

A set of matrices

Given M_n, define $A_n(k)$

$$A_n(k) = \begin{bmatrix} M_n(k) & 0 \\ 0 & 0 \end{bmatrix}.$$

$M_n(k)$ is the $k \times k$ minor of M_n.
For $t \in [0,1]$ we define $s_n(z,t)$ such that when $t = \frac{k}{n} = t_n(k)$

$$s_n(z,t) := \frac{1}{n} trace([A_n(k) - zI]^{-1}).$$

Towards the derivative

$$\frac{s_n(z, t_n(k+1)) - s_n(z, t_n(k))}{t_n(k+1) - t_n(k)}$$

$$= trace([A_n(k+1) - zI]^{-1}) - trace([A_n(k) - zI]^{-1}).$$

Using the tricks

$$= trace([M_n(k+1) - zI]^{-1}) - trace([M_n(k) - zI]^{-1}) + \frac{1}{z}$$

and

$$trace([M_n(k+1) - zI]^{-1}) - trace([M_n(k) - zI]^{-1})$$

$$= \frac{1 + v_n^\top(k)\,(M_n(k) - zI)^{-2}\,v_n(k)}{\alpha_n(k) - v_n^\top(k)\,(M_n(k) - zI)^{-1}\,v_n(k)}.$$

Elements of the equation

for $j = 1, 2$ the limit of

$$T_k^{(j)}(n) = v_n^\top(k)\,(M_n(k) - zI)^{-j}\,v_n(k),$$

$$= \gamma_n^\top(k+1)[\gamma_n(1), \ldots, \gamma_n(k)]\,(M_n(k) - zI)^{-j} \begin{bmatrix} \gamma_n^\top(1) \\ \vdots \\ \gamma_n^\top(k) \end{bmatrix} \gamma_n(k+1).$$

$\gamma_n(k)$ is the k column of X_n.
The all the variables in $\gamma_n(k+1)$ are independent from the rest of $T_k^{(j)}(n)$. Define $\mathcal{F}_{k,n}$ to be the information inside $T_k^{(j)}(n)$ without considering $\gamma_n(k+1)$.

Limits of the elements 1

Then

$$E[T_k^{(j)}(n)|\mathcal{F}_{k,n}] = \frac{1}{n} Trace\left(\left(M_n(k) - zI\right)^{-j} M_n(k)\right)$$
$$= \frac{1}{n}\sum_{i=1}^{k} \frac{\lambda_i(k)}{(\lambda_i(k) - z)^j},$$

By Helly's theorem there exists a subsequence k_n such that

$$\frac{1}{n}\sum_{i=1}^{k_n} \frac{1}{(\lambda_i(k_n) - z)} \to s(z,t)$$

Limits of the elements 2

by adding and substracting z we obtain that

$$\lim_n E[T^{(1)}_{k_n}(n)|\mathcal{F}_{k_n,n}] = 1 + zs(z,t)$$

and

$$\lim_n E[T^{(2)}_{k_n}(n)|\mathcal{F}_{k_n,n}] = z\frac{\partial s(z,t)}{\partial z} + s(z,t).$$

Moreover, also

$$\lim_n Var[T^{(j)}_{k_n}(n)|\mathcal{F}_{k_n,n}] = 0.$$

The final equation

We end up with

$$s_t(z,t) = -\frac{s_z(z,t)}{1+s(z,t)},$$

and the initial condition is clearly $s(z,0) = -\frac{1}{z}$.
To find $S(z) = s(z,1)$, by the method of characteristics

$$S(z) = -\frac{1}{2} + \frac{\sqrt{(z-2)^2-4}}{2z}.$$

This Stieltjes transform comes from the density

$$f_{MP}(x) = \frac{\sqrt{(4-x)x}}{2\pi x}, \ x \in [0,4].$$