Random Matrix Theory: Lecture 3

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CINVESTAV

Density of the matrix

Consider the Wigner GOE model $M_n = \frac{1}{2}(X_n + X_n^\top)$ Then,

$$P(M_n \in dx) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-x_i^2/2} \prod_{1 \le i < j \le n} \frac{1}{\sqrt{\pi}} e^{-x_{ij}^2}.$$

or

$$P(M_n \in dx) = c_n \prod_{i,j=1}^{n} e^{-x_{ij}^2/2} = c_n e^{-\frac{1}{2}\sum_{i,j=1}^{n} x_{ij}^2} = c_n e^{-\frac{Trace(M_n^2)}{2}},$$

with $x_{ij} = x_{ji}$.
S_n the space of symmetrix $n \times n$ matrices; a manifold of dimension $N = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$.
Represent M_n as a vector $\bar{v} = (v_1, \ldots v_N) \in \mathbf{R}^N$, using the map

$$\phi(\bar{v}) = M_n,$$

where the first n elements of $\bar{v}$ form the diagonal of M_n, then $v_{n+1}, \ldots, v_{n+(n-1)}$ form the first row (and also the first column), and so forth.

Joint density of a long vector

the joint probability density of $\bar{v}$ is

$$f(\bar{v}) = c_n e^{-\frac{1}{2} Trace(\phi(\bar{v})^2)}$$

From the spectral decomposition $M_n = U^\top \Lambda U$, where U is unitary and Λ is the diagonal matrix with the eigenvalues $\lambda_1 \leq \ldots \leq \lambda_n$.
Given the vector $\bar{w} = (\lambda_1, \ldots, \lambda_n, u_1, \ldots, u_m)$, where $m = N - n$,

$$\psi(\bar{w}) = M_n = U^\top \Lambda U,$$

where $\bar{\lambda} = (\lambda_1, \ldots, \lambda_n)$ form the diagonal matrix Λ, and
$\bar{u} = (u_1, \ldots, u_m)$ form the matrix U.
Our aim is to find the joint probability density $h_n(\bar{\lambda})$ of $\bar{\lambda}$

Let $g(\bar{w})$ be the density of $\bar{w}$
From the spectral decomposition we can see that there is a differentiable map F from $\bar{v}$ to $\bar{w}$. Then, the probability density of $\bar{w}$ is given by $g(\bar{w}) = f(F^{-1}(\bar{w}))|\det J(\bar{w})|$,
where $J(\bar{w})$ is the Jacobian matrix of F.

Notice is that $f(F^{-1}(\bar{w})) = c_n e^{-\frac{\lambda_1^2+\ldots+\lambda_n^2}{2}}$.
So, our task now is to find the Jacobian

$$J = \begin{bmatrix} \frac{dv_1}{d\lambda_1} & \cdots & \frac{dv_1}{d\lambda_n} & \frac{dv_1}{du_1} & \cdots & \frac{dv_1}{du_m} \\ \frac{dv_2}{d\lambda_1} & \cdots & \frac{dv_2}{d\lambda_n} & \frac{dv_2}{du_1} & \cdots & \frac{dv_2}{du_m} \\ & & \vdots & & & \\ \frac{dv_N}{d\lambda_1} & \cdots & \frac{dv_N}{d\lambda_n} & \frac{dv_N}{du_1} & \cdots & \frac{dv_N}{du_m} \end{bmatrix}.$$

derivatives in the manifold

Use $M_n = U^\top \Lambda U$ to find the derivatives. All the information is inside $\{\frac{dM_n}{d\lambda_i}, \ \frac{dM_n}{du_j} : i = 1, \dots, n; j = 1, \dots, m\}$.
We first have that

$$\frac{dM_n}{d\lambda_i} = U^\top \frac{d\Lambda}{d\lambda_i} U \text{ and } \frac{dM_n}{du_j} = U^\top \Lambda \frac{dU}{du_j} + \frac{dU^\top}{du_j} \Lambda U.$$

$\frac{d\Lambda}{d\lambda_i}$ is the matrix full of zeros and only one 1 in the (i, i)-entry.
And

$$\frac{dM_n}{du_j} = U^\top \left(\Lambda \frac{dU}{du_j} U^\top + U \frac{dU^\top}{du_j} \Lambda \right) U$$

Also from $I = UU^\top$, we have $0 = U \frac{dU^\top}{du_j} + \frac{dU}{du_j} U^\top$.
Hence

$$\frac{dM_n}{du_j} = U^\top \left(D_j \Lambda - \Lambda D_j \right) U,$$

where $D_j = U \frac{dU^\top}{du_j}$ depends solely on the variables $u's$.

It ocurrs that $|det(J)| = |det(J')|$, where J' is the $N \times N$ matrix with the columns

$$c'_i = \phi^{-1}\left(\frac{d\Lambda}{d\lambda_i}\right), i = 1, \ldots, n;$$

and

$$c'_{n+j} = \phi^{-1}\left(D_j\Lambda - \Lambda D_j\right), j = 1, \ldots, m.$$

Notice that $D_r\Lambda - \Lambda D_r = [d_{ij}(r)(\lambda_j - \lambda_i)]_{i,j=1}^n$, where $d_{ij}(r)$ are the entries of D_r.

The matrix J' is

$$\begin{bmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ & & \vdots & & & & \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & d_{1,2}(1)(\lambda_2 - \lambda_1) & \dots & d_{1,2}(m)(\lambda_2 - \lambda_1) \\ 0 & 0 & \dots & 0 & d_{1,3}(1)(\lambda_3 - \lambda_1) & \dots & d_{1,3}(m)(\lambda_3 - \lambda_1) \\ & & \vdots & & & & \\ 0 & 0 & \dots & 0 & d_{n-1,n}(1)(\lambda_n - \lambda_{n-1}) & \dots & d_{n-1,n}(m)(\lambda_n - \lambda_{n-1}) \end{bmatrix}.$$

Therefore, $|det(J)| = |det(J')| = \prod_{1\leq i<j\leq n}(\lambda_j - \lambda_i)H(U)$, where $H(U)$ is independent of λ_i.

Thus, the final conclusion is that for the GOE

$$h_n(\bar{\lambda}) = \frac{1}{C_n} e^{-\frac{\lambda_1^2+\ldots+\lambda_n^2}{2}} \prod_{1\leq i<j\leq n} (\lambda_j - \lambda_i),$$

over the set $A_n = \{\lambda_1 \leq \ldots \leq \lambda_n\} \subset \mathbb{R}^n$.