Random Matrix Theory: Lecture 4

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CINVESTAV

For the GUE

Density of eigenvalues

$$h(\bar{\lambda}) = \frac{1}{C_n} e^{-\frac{\lambda_1^2+\ldots+\lambda_n^2}{2}} \prod_{1\leq i<j\leq n} (\lambda_j - \lambda_i)^2, \tag{1}$$

on $A_n = \{\lambda_1 \leq \ldots \leq \lambda_n\} \subset \mathbb{R}^n$
Or

$$h(\bar{\lambda}) = \frac{1}{Z_n} e^{-(V(\lambda_1)+\ldots+V(\lambda_n))} \prod_{1\leq i<j\leq n} (\lambda_j - \lambda_i)^2, \tag{2}$$

on $\mathbb{R}^n$, where $Z_n = n! C_n$ and $V(x) = \frac{x^2}{2}$.

Vandermonde

The very first thing to know is the so-called Vandermonde determinant:

$$\Delta_n(\bar{\lambda}) := \prod_{1 \le i < j \le n} (\lambda_j - \lambda_i) = detM,$$

where $M = [\lambda_i^{j-1}]_{i,j=0}^n$.
Consider n polynomials $p_1(x), \ldots, p_n(x)$,

$$p_i(x) = \sum_{j=1}^{n} a_{i,j} x^{j-1}$$

If $A = [a(i,j)]_{i,j=1}^n$,
since $det(MA) = det(M)det(A)$ and $MA = [p_j(\lambda_i)]_{i,j=1}^n$. Then

$$\Delta_n(\bar{\lambda}) = det[p_i(\lambda_j)]_{i,j=1}^n / det(A).$$

if p_i is of order i and monic, $det(A) = 1$.
Then

$$h(\bar{\lambda}) = \frac{1}{Z_n}\left(e^{-(V(\lambda_1)+\ldots+V(\lambda_n))/2} det([p_i(\lambda_j)])\right)^2$$

However $e^{-(V(\lambda_1)+\ldots+V(\lambda_n))/2}$ is a determinant of B (diagonal).
Thus

$$\begin{aligned} Z_n h(\bar{\lambda}) &= (det(B) det([p_i(\lambda_j)]))^2 \\ &= det(B[p_i(\lambda_j)](B[p_i(\lambda_j)])^T) \end{aligned}$$

Since $B[p_i(\lambda_j)] = [P_i(\lambda_j)]_{i,j=1}^n$, with $P_i(x) = e^{-V(x)/2} p_i(x)$, we end up with

$$h(\bar{\lambda}) = \frac{1}{Z_n} det([K_n(\lambda_i, \lambda_j)]_{i,j=1}^n),$$

where $K_n(x, y) = \sum_{k=1}^n P_k(x) P_k(y)$.

Hermite polynomials

take p_i/γ_i orthonormal with respect to $e^{-V(x)/2}$, i.e. the probabilistic Hermite polynomials. Then

$$h_n(\bar{\lambda}) = \frac{1}{n!} det([K_n(\lambda_i, \lambda_j)]_{i,j=1}^n),$$

Lemma

$$\int_{\mathbb{R}^{n-k}} h_n(\bar{x}) dx_{k+1} \dots dx_n$$

$$= \frac{(n-k)!}{n!} det([K_n(x_i, x_j)]_{i,j=1}^k.$$

Probabilities

Let us now calculate

$$A_n(\theta) = P(\text{there are no eigenvalues inside } (-\theta, \theta)),$$

or

$$F_n(\theta) = P(\lambda_{max} < \theta).$$

Fredholm determinant

Let $I_\theta(x)$ be the indicator function of the set $\{|x| \leq \theta\}$.
From the fact that the variables in the density h_n are exchangable

$$\begin{aligned}
A_n(\theta) &= P(|\lambda_1| > \theta, \ldots, |\lambda_n| > \theta) \\
&= \int_{\mathbb{R}^n} \prod_{i=1}^{n} (1 - I_\theta(x_i)) h_n(\bar{x}) dx_1 \ldots dx_n \\
&= 1 + \sum_{k=1}^{n} (-1)^k \binom{n}{k} \int_{\mathbb{R}^k} \prod_{i=1}^{k} I_\theta(x_i) \int_{\mathbb{R}^{n-k}} h_n(\bar{x}) dx_1 \ldots dx_n \\
&= 1 + \sum_{k=1}^{n} (-1)^k \binom{n}{k} \frac{(n-k)!}{n!} \int_{-\theta}^{\theta} \ldots \int_{-\theta}^{\theta} det[K_n(x_i, x_j)]_{i,j=1}^{k} dx_1 \\
&= det[I - K_n],
\end{aligned}$$

where the last expression represents the Fredholm determinant of the operator $I - K_n$, and K_n is an integral operator with kernel K_n acting on the Hilbert space $L_2((-\theta, \theta))$.

again..

Since

$$F_n(\theta) = P(\lambda_1 \leq \theta, \dots, \lambda_n \leq \theta),$$

by similar calculations as before we find that

$$F_n(\theta) = det[I - K_n],$$

but in this case the Hilbert space is $L_2((\theta, \infty))$.