Random Matrix Theory: Lecture 5

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CINVESTAV

Point process

Definition

A **point process** over a space Λ is an random object X such that $X(D)$ is the number of points in a set $D \subset \Lambda$, which is random variable on $\mathbb{N}$. It is said that X has **intensity funcions** (or **correlation functions**) $\rho_k : \Lambda^k \to \mathbb{R}$, with respecto to μ, if

$$E\left[\prod_{i=1}^{k} X(D_i)\right] = \int_{D_1} \cdots \int_{D_k} \rho_k(x_1, \ldots, x_k)\mu(dx_k) \ldots \mu(dx_1),$$

where $D_i \subset \Lambda$ are disjoint sets.

Example

Let $V = (X_1, \ldots, X_n)$ be a random vector on $\mathbb{R}^n$, such that the variables are exchangeable, i.e. any permutation of the variables gives the same probability distribution. Let $h_n(x_1, \ldots, x_n)$ be its probability density. Then, $X(D) = \sum_{i=1}^{n} \delta_{X_i}(D)$ is a point process on $\mathbb{R}$, w.r.t. the Lebesgue measure, and its correlation functions are

$$\rho_k(x_1, \ldots, x_k) = \frac{n!}{(n-k)!} \int_{\mathbb{R}^{n-k}} h_n(x_1, \ldots, x_n) dx_{k+1} \ldots dx_n, \quad (1)$$

with $k = 1, \ldots, n$.

Determinantal process

Definition

A **determinantal point process** is such that

$$\rho_k(x_1, \dots, x_k) = det[K(x_i, x_j)]_{i,j=1}^k,$$

where $K : \Lambda \times \Lambda \to \mathbb{C}$ is measurable, and it is called the **kernel** of the process.

Bergman point process

Example

See Peres and Virag (2005). Let us present this first interesting example, the where the points are zeros of a random function. Define $f(z) = \sum_{n=1}^{\infty} a_n z^n$ where a_n are i.i.d. Gaussian r.v.s over $\mathbb{C}$. Then

1. The radius of convergence is 1 almost surely.
2. The set $\mathcal{Z} = \{z \in \mathbb{C} : f(z) = 0\}$ is almost surely infinite but countable.
3. $X(D) = |D \bigcap \mathcal{Z}|$ is a determinantal point process over $\mathbb{D}$, w.r.t. the Lebesgue measure, with kernel

$$K(x, w) = \frac{1}{\pi(1 - z\bar{w})^2},$$

which is called the Bergman kernel.

from Ginibre ensemble

Example

See Ginibre (1965). Let M be a $n \times n$ random matrix where the entries are independent Gaussian r.v.s. Then, the eigenvalues of M form a determinantal process over $\mathbb{C}$, w.r.t. $\mu(dz) = e^{-|z|^2} dz/\pi$, with kernel

$$K(z, w) = \sum_{k=0}^{n-1} \frac{(z\bar{w})^k}{k!}.$$

Example

Take $\Lambda = \mathbb{R}$. It turns out that the kernel

$$K(x, y) = \frac{sin(\pi(x - y))}{\pi(x - y))}$$

defines a determinantal process, w.r.t. the Lebesgue measure. This process arises in the limit in different physical and mathematical situations, that is why it is said to be a universal object.

Let X_n be the point process from the eigenvalues of the GUE ensemble.
From one Lemma we can see that X_n has correlation functions given by

$$\begin{aligned}\rho_k(x_1,\dots,x_k) &= \frac{n!}{(n-k)!}\int_{\mathbb{R}^{n-k}} h_n(x_1,\dots,x_n)dx_{k+1}\dots dx_n \\ &= det([K_n(x_i,x_j)]_{i,j=1}^k\end{aligned}$$

with $k = 1,\dots,n$.

Christoffer-Darboux

Accoding to the Christoffer-Darboux indentity

$$
\begin{aligned}
K_{n+1}(x,y) &= e^{-(V(x)+V(y))/2}\sum_{k=0}^{n}\frac{h_k(x)h_k(y)}{\gamma_k^2} \\
&= e^{-(V(x)+V(y))/2}\frac{1}{\sqrt{2\pi}n!}\frac{h_{n+1}(x)h_n(y)-h_n(x)h_{n+1}(y)}{x-y} \\
&= e^{-\frac{1}{2}((x/\sqrt{2})^2+(y/\sqrt{2})^2)}\frac{2^{-n}}{2\sqrt{\pi}n!}\times \\
&\quad \frac{H_{n+1}(x/\sqrt{2})H_n(y/\sqrt{2})-H_n(x/\sqrt{2})H_{n+1}(y/\sqrt{2})}{x-y},
\end{aligned}
$$

Asymptotics

$$e^{-(x/\sqrt{2})^2/2}H_{n+1}(x/\sqrt{2}) \sim \frac{2^{n+1}}{\sqrt{\pi}}\Gamma\left(\frac{n+2}{2}\right)\cos\left(\sqrt{n+1}x - \frac{(n+1)\pi}{2}\right),$$

and

$$e^{-(y/\sqrt{2})^2/2}H_n(y/\sqrt{2}) \sim \frac{2^n}{\sqrt{\pi}}\Gamma\left(\frac{n+1}{2}\right)\cos\left(\sqrt{n}y - \frac{n\pi}{2}\right),$$

the Sine kernel

Take $x = \frac{u}{\sqrt{n}}$ and $y = \frac{v}{\sqrt{n}}$

$$
\begin{aligned}
K_{n+1}(x,y) \quad &\sim \quad \frac{2^{-2k}}{2\sqrt{\pi}(2k)!}\frac{2^{2k+1}}{\sqrt{\pi}}\frac{2^{2k}}{\sqrt{\pi}}\sqrt{\pi}\frac{(2k)!}{2^{2k}k!}k! \\
&\quad \frac{\cos(u)\sin(v)-\sin(u)\cos(v)}{u-v}\sqrt{n} \\
&= \quad \frac{\sqrt{n}}{\pi}\frac{\cos(u)\sin(v)-\sin(u)\cos(v)}{u-v} \\
&= \quad \frac{\sqrt{n}}{\pi}\frac{\sin(u-v)}{u-v}
\end{aligned}
$$

Then in the limit the set of eigenvalues form the

Sine point process.