Random Matrix Theory: Lecture 6

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CINVESTAV

$$A_n(t) = P(\text{there are no eigenvalues inside } (-t,t)) =$$

$$1 + \sum_{k=1}^{n} \frac{(-1)^k}{k!} \int_{-t}^{t} \dots \int_{-t}^{t} det[K_n(x_i, x_j)]_{i,j=1}^k dx_1 \dots dx_k$$

Rescaling with $1/\sqrt{n}$, as $n \to \infty$, $A_n(\theta)$ converges to

$$A(t) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{-t}^{t} \dots \int_{-t}^{t} det[k(x_i, x_j)]_{i,j=1}^k dx_1 \dots dx_k,$$

where

$$k(x,y) = \frac{\sin(x-y)}{\pi(x-y)}.$$

Fredholm Determinant

Consider the operator K defined through a kernel k

$$Kf(x) = \int_s^t k(x,y)f(y)dy,$$

acting on function in $L_2([s,t])$.
In the pursuit of solving the equation $f - Kf = h$, Fredholm proposed studying the expression

$$D \ = \ 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_s^t \cdots \int_s^t det[k(x_i,x_j)]_{i,j=1}^k dx_1 \dots dx_k.$$

This is now called the Fredholm determinant of K, denoted $det(I-K)$, it is finite!.

The resolvent

It turns out that one can find the inverse of the operator $I - K$. Using the notation of Fredholm, define

$$k\begin{pmatrix} x_1 & \dots & x_n \\ y_1 & \dots & y_n \end{pmatrix} = det[k(x_i, y_j)]_{i,j=1}^n.$$

Now, we can define the so-called resolvent kernel

$$r(x,y) := \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k!} \int_s^t \dots \int_s^t k\begin{pmatrix} x & x_1 & \dots & x_k \\ y & x_1 & \dots & x_k \end{pmatrix} dx_1 \dots dx_k.$$

Lemma

It holds the following two identities

$$\begin{aligned} r(x,y) - Dk(x,y) &= \int_s^t k(x,z) r(z,y) dz \\ &= \int_s^t r(x,z) k(z,y) dz. \end{aligned}$$

Derivative of A

Notice that $\frac{dA_n(t)}{dt} =$

$$-2\sum_{k=1}^{n}\frac{(-1)^k}{(k-1)!}\int_{-t}^{t}\dots\int_{-t}^{t}K_n\begin{pmatrix} t & x_1 & \dots & x_{k-1} \\ t & x_1 & \dots & x_{k-1}\end{pmatrix}dx_1\dots dx_{k-1}$$

where

$$K_n\begin{pmatrix} x_1 & \dots & x_n \\ y_1 & \dots & y_n\end{pmatrix} = det[K_n(x_i, y_j)]_{i,j=1}^{n}$$

Hence

$$\frac{dA}{dt} = \frac{dD}{dt} = -2r(t,t),$$

where r is the resolvent kernel associated to k.
If $R(x,y) = r(x,y)/D$,

$$\frac{dlogA}{dt} == -2R(t,t), \qquad (1)$$

Aim: characterize $\sigma(t) = tR(t,t)$.

Towards a differential equation

Notation: $H = \int h(x,y)$ is the integral operator with kernel h. Let $T = \int R(x,y)$. • the identity can be seen as an integral operator with kernel the delta Dirac

$$h(x) = \int \delta(x-y)h(y)dy$$

If $f(x) = \sin(x)/\sqrt{\pi}$ and $g(x) = \cos(x)/\sqrt{\pi}$, then

$$k(x,y) = \frac{f(x)g(y) - f(y)g(x)}{x-y}.$$

Also we define

$$Q(x) = (I+T)f(x) = f(x) + \int_{-t}^{t} R(x,y)f(y)dy \tag{2}$$

and

$$P(x) = (I+T)g(x) = g(x) + \int_{-t}^{t} R(x,y)g(y)dy. \tag{3}$$

Step 1: The conmmutator

$[T_1, T_2] := T_1 T_2 - T_2 T_1$

Proposition Let M be the multiplicacion opertor by the argument, that is $Mf(x) = xf(x)$. Then $[M, H] = \int (x - y)h(x, y)$.

Proposition Suppose that $I - T_2$ is invertible, then it holds

$$[T_1, (I - T_2)^{-1}] = (I - T_2)^{-1}[T_1, T_2](I - T_2)^{-1}.$$

Corollary $[M, I + T] = (I + T)[M, K](I + T)$.

Theorem It holds that

$$R(x, y) = \frac{Q(x)P(y) - P(x)Q(y)}{x - y}.$$

Proposition

$$R(-t, t) = \frac{Q(t)P(t)}{t}.$$

Step 2: x, y derivatives

We work on the interval $[t_1, t_2]$.
Lemma It holds

$$-\left(\frac{dR}{dx}+\frac{dR}{dy}\right)=R(x,t_2)R(t_2,y)-R(x,t_1)R(t_1,y),$$

Proposition

$$Q'(x)=P(x)+R(x,t_1)Q(t_1)-R(x,t_2)Q(t_2)$$

and

$$P'(x)=-Q(x)+R(x,t_1)P(t_1)-R(x,t_2)P(t_2).$$

Hence

$$R(t,t)=P^2(t)+Q^2(t)-2\frac{Q^2(t)P^2(t)}{t}$$

Step 3: t derivatives

Lemma

$$\frac{dR}{dt}(x,y) = R(x,t)R(t,y) + R(x,-t)R(-t,y).$$

Proposition

$$\frac{dR}{dt}(t,t) = 2R^2(-t,t).$$

Proposition

$$\frac{dQ(t)}{dt} = P(t) - 2\frac{P(t)Q^2(t)}{t}$$

and

$$\frac{dP(t)}{dt} = Q(t) + 2\frac{Q(t)P^2(t)}{t}.$$

Finally...

Theorem

σ satisfies the Painlevé V differential equation

$$(t\sigma''(t))^2 = 8\left((\sigma'(t))^2 + 2t\sigma'(t) - 2\sigma(t)\right)\left(t\sigma'(t) - \sigma(t)\right).$$

Tracy-Widom distrubution (F_2)

Using the other asymtotics for the Hermite polynomials, we can study the limit behaviour of eigenvalues around the edge of the semi-circle law, that around 2 or -2.
It turns out that

$$\frac{1}{n^{1/6}} K_n \left(2\sqrt{n} + \frac{x}{n^{1/6}}, 2\sqrt{n} + \frac{y}{n^{1/6}} \right)$$

$$\sim \frac{Ai(x)Ai'(y) - Ai(y)Ai'(x)}{x - y}$$

And the distribution of the largest eigenvalue is a Fredholm determinant with this kernel:

$$\lim_{n\to\infty} n^{2/3}(\lambda_n/\sqrt{n} - 2) \sim F_2$$

F_2, called the Tracy-Widom distribution, is determined by differential equation $\sigma''(t) = t\sigma(t) + 2\sigma^3(t)$, $\sigma(t) \sim Ai(t)$ as $t \to \infty$